

APPLIED MATHEMATICS-III

Q. 1) a) *If* 05

$$\int_0^{\infty} e^{-2t} \cdot \sin(t + \alpha) \cdot \cos(t + \alpha) dt = \frac{3}{8}, \text{ then find } \alpha.$$

Ans: $L(\sin(t+\alpha) \cdot \cos(t-\alpha)) = L\left(\frac{\sin 2t + \sin 2\alpha}{2}\right) = \frac{1}{2} \left[\frac{2}{s^2 + 4} + \sin 2\alpha \cdot \frac{1}{s} \right] = \frac{3}{8}$ 01

$$\left[\frac{2}{s^2 + 4} + \sin 2\alpha \cdot \frac{1}{s} \right] = \frac{3}{4}$$
 02

For s = 2, $\left[\frac{2}{4+4} + \sin 2\alpha \cdot \frac{1}{2} \right] = \frac{3}{4}$ 03

So, $\sin 2\alpha = 1$

$2\alpha = \pi/2$ 05

$\alpha = \pi/4.$

b) *Prove that every Hermitian matrix A can be express as B + iC, where B is real symmetric and C is real skew symmetric matrix.* 05

Ans: $A = B + iC$ 01

$$= \frac{1}{2} \{(A + \bar{A})\} + i \frac{1}{2i} \{(A - \bar{A})\}$$

If Z & \bar{Z} be complex and complex conjugates of each other then

$\frac{1}{2} (z + \bar{z})$ and $\frac{1}{2i} (z - \bar{z})$ are real 02

$B^T = \frac{1}{2} \{(A + \bar{A})\}^T = \frac{1}{2} \{(A^T + \bar{A}^T)\} = \frac{1}{2} \{(A^T + A^{\theta T})\}$ since A is Hermitian is given.

$$= \frac{1}{2} \{(A + \bar{A})\} = B$$
 03

So, B is symmetric. 05

Similarly we can prove, $C^T = -C$ i.e. C is skew – symmetric.

c) *If f(z) is an analytic function, prove that* 05

Ans: $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] \log |f'(z)| = 0$ 01

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$$F(z) = u + iv$$

$$F'(z) = u_x + i v_x$$

$$|f'(z)| = \sqrt{u_x^2 + v_x^2}$$

$$\log|f'(z)| = \log\sqrt{u_x^2 + v_x^2}$$

$$\log|f'(z)| = \frac{1}{2}\log|f'(z)|^2$$

$$\text{But, } |f'(z)|^2 = f'(z) \cdot \overline{f'(z)}$$

$$\log|f'(z)| = \frac{1}{2}\log[f'(z) \cdot \overline{f'(z)}]$$

$$= \frac{1}{2}[\log f'(z) + \log \overline{f'(z)}]$$

Now,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 4\frac{\partial^2}{\partial z\partial\bar{z}}\log|f'(z)|$$

$$= 4\frac{\partial^2}{\partial z\partial\bar{z}}\left[\frac{1}{2}\log|f'(z)| + \frac{1}{2}\log|f'(\bar{z})|\right]$$

$$= 2\frac{\partial^2}{\partial z\partial\bar{z}}[\log|f'(z)| + 2\frac{\partial^2}{\partial z\partial\bar{z}}\log|f'(\bar{z})|]$$

$$= 2\frac{\partial}{\partial z}\left(\frac{\partial}{\partial\bar{z}}\log|f'(z)|\right) + 2\frac{\partial}{\partial\bar{z}}\left(\frac{\partial}{\partial z}\log|f'(\bar{z})|\right)$$

$$= 2\frac{\partial}{\partial z}(0) + 2\frac{\partial}{\partial\bar{z}}(0) = 0$$

d) Evaluate $\int_c (z^2 + z) dz$ along the line joining $(1, -1), (2, 3)$

Ans:

$$\frac{y+1}{-1-1} = \frac{x-3}{3-2}$$

$$\frac{y+1}{-2} = \frac{x-3}{1}$$

$$Y + 1 = -2(x - 3)$$

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$$Y + 1 = -2x + 6$$

$$Y + 2x = 5$$

$$Y = 5 - 2x$$

$$Dy = -2 dx$$

02

$$Dz = dx + idy = dx = 2idx = (1-2i) dx$$

$$I = \int_1^2 (x + iy)^2 + (x + iy)(dx + idy)$$

04

$$= \int_1^2 (x^2 - y^2 + 2ixy) + (x + iy)(dx + idy)$$

Sub. $Y = 5 - 2x$ we get,

$$I = -75/6 + 32i/3$$

05

Q. 2) a) Find the image of the triangle whose vertices are $i, 1+i, 1-i$ under the transformation $w = 3z + 4 - 2i$.

06

Ans:

$$U + iv = 3(x+iy) + 4 - 2i$$

02

$$= 3x + i3y + 4 - 2i$$

$$(3x+4) + i(3y-2)$$

Here $u = 3x + 4$ and $v = 3y - 2$.

06

Solving we get vertices of triangle are:

$$(4,1) (7,1) \text{ and } (7,-5)$$

b)

Find the inverse Laplace transform of $\cot^{-1}\left(\frac{2}{s^2}\right)$.

07

Ans: $\cot^{-1} s = \frac{\pi}{2} - \tan^{-1} s$

01

So,

$$\cot^{-1} \frac{2}{s^2} = \frac{\pi}{2} - \tan^{-1} \frac{2}{s^2}$$

Let

$$\varphi(s) = \frac{\pi}{2} - \tan^{-1} \frac{2}{s^2}$$

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$$\varphi'(s) = \frac{1}{1+\frac{4}{s^2}} \left(\frac{-4}{s^2}\right) = \frac{-4s}{s^2+4}$$

$$\text{So, } \frac{1}{t} L^{-1}(\varphi'(s)) = \frac{-4}{t} L^{-1}\left(\frac{s}{s^2+4}\right)$$

02

$$= \frac{-4}{t} L^{-1}\left[\frac{s}{(s^2+2)^2 - 2s^2}\right]$$

$$= \frac{-4}{t} L^{-1}\left[\frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}\right]$$

03

$$= \frac{-1}{t} L^{-1}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$$= \frac{-1}{t} e^{t} L^{-1}\left(\frac{1}{s^2+1}\right) - e^{-t} L^{-1}\left(\frac{1}{s^2+1}\right)$$

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$$= -1/t (e^t \sin t - e^{-t} \sin t)$$

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$$= (-2 \sin t - \sin ht) / t.$$

- c) Find non singular matrices P and Q such that PAQ is in normal form. Also find rank of A, and A⁻¹.

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Where

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Ans:

$$A = I_3 \quad A \quad I_3$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

01

R13

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$R2 - 3R1, R3 - 2R1$$

02

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C2 - 2C1, C3 - 3C1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 7 \\ 0 & -3 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$SO, \zeta(A) = 2$$

$$A^{-1} = \frac{1}{6} \begin{bmatrix} -1 & 3 & -1 \\ -7 & 3 & 5 \\ 5 & -3 & -1 \end{bmatrix}$$

05

Q.3) a) Show that the vectors x_1, x_2, x_3 are linearly independent and the vector x_4 depends upon them, where, $x_1 = (1, 2, 4)$, $x_2 = (2, -1, 3)$, $x_3 = (0, 1, 2)$, $x_4 = (-3, 7, 2)$.

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Ans:

$$Ax_1 + b x_2 + c x_3 = 0$$

$$A(1, 2, 4) + b(2, -1, 3) + c(0, 1, 2) = 0$$

01

$$A + 2b + 0c = 0$$

$$2a - b + c = 0$$

$$4a + 3b + 2c = 0$$

$$AX = 0$$

02

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R2 - 2R1, R3 - 4R1$$

04

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$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 1 \\ 0 & -5 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

05

R3-R2

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

SO, X1, X2, X3 are linearly independent.

06

But for X4 = (-3,7,2) given system becomes dependent.

So, X4 depend on them.

b) Find the inverse Laplace transform by using convolution theorem of the

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function $\frac{1}{s\sqrt{s+4}}$.

Ans:

$$L^{-1}\left(\frac{1}{(s+4)^{1/2}}\right) = e^{-4u} \frac{u^{-1/2}}{\sqrt{1/2}}$$

01

And $L^{-1}\left(\frac{1}{s}\right) = 1$

02

$$L^{-1}(\phi_1 \phi_2) = L^{-1}\left(\frac{1}{s} \cdot \frac{1}{(s+4)^{1/2}}\right)$$

$$= \int_0^x \frac{e^{-4u} u^{-1/2}}{\sqrt{\pi}} (1) du$$

04

Put $4u = x^2$

$$du = \frac{x dx}{2}, \sqrt{u} = x/2$$

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$$L^{-1}\left(\frac{1}{s} \cdot \frac{1}{(s+4)^{1/2}}\right) = \int_u^{2\sqrt{t}} \frac{e^{-x^2}}{\sqrt{\pi}} dx$$

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$$= 1/2 \operatorname{erf}(2\sqrt{t})$$

c) State Residue theorem. Hence evaluate

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$$\int_0^{2\pi} \frac{d\theta}{3 - 2 \cos \theta + \sin \theta}$$

Ans:

Statement: If f(Z) is an analytic function everywhere except some finite

01

number of points then

$$\int f(z)dz = 2\pi i(\text{sum of the residues})$$

Let $I = \int_C \frac{dz}{iz^3 - 2z^2 + iz}$

03

$$= \int_C \frac{dz}{iz^3 - 2iz^2 + iz + (z^2 - 1)}$$

05

$$= \int_C \frac{dz}{iz^3 - 2iz^2 - 2i + (z^2 - 1)}$$

$$= \int_C \frac{1}{(1 - 2i)z^2 + 6iz - (1 + 2i)} dz$$

So, $I = \pi(2+i)$

07

Q.4) a)

Find the analytic function $f(z) = u + iv$ such that $u+v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

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Ans:

$F(z) = u + iv$

02

$if(z) = ui - v$

$(1+i)f(z) = (u-v) + i(u+v)$

$= U + iV$

$V = u+v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

$V_x = \frac{2\cosh 2y \cos 2x - 2}{(\cosh 2y - \cos 2x)^2}$

04

$V_y = \frac{-2\sinh 2y \sin 2x}{(\cosh 2y - \cos 2x)^2}$

By Milne Thompson's method,

$(1+i) f'(z) = V_y + iV_x$

$= \square 1(z,0) + i\square 2(z,0)$

$(1+i) f(z) = \int 0 + i \frac{2(\cos 2x - 1)}{(1 - \cos 2x)^2} dz$

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$$= -2i \int \frac{dz}{1 - \cos 2z}$$

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$$= -i \int \operatorname{cosec}^2 z \, dz$$

$$= I \cot z + c'$$

$$\text{So, } f(z) = \frac{1}{1+i} \cot z + c.$$

06

b) Express the following function as Heaviside's unit step function and find its Laplace transform

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$$f(t) = \cos t, 0 < t < \pi$$

$$= \cos 2t, \pi < t < 2\pi$$

$$= \cos 3t, t > 2\pi$$

Ans:

$$F(t) = \cos t (H(t) - H(t-\pi)) + \cos 2t [H(t-\pi) - H(t-2\pi)] + \cos 3t [H(t-2\pi)]$$

01

$$= \cos t H(t) + (\cos 2t - \cos t) H(t-\pi) + (\cos 3t - \cos 2t) H(t-2\pi)$$

03

$$\text{So, } Lf(t) = L(\cos t) + e^{-\pi s} L(\cos 2(t+\pi) - \cos(t+\pi)) + e^{-2\pi s} L[\cos 3(t+2\pi) - \cos 2(t+2\pi)]$$

04

$$= L(\cos t) + e^{-\pi s} (\cos 2t + \cos t) + e^{-2\pi s} (\cos 3t - \cos 2t)$$

06

$$= \frac{s}{s^2+1} + e^{-\pi s} \left[\frac{s}{s^2+4} + \frac{s}{s^2+1} \right] + e^{-2\pi s} \left[\frac{s}{s^2+9} - \frac{s}{s^2+4} \right]$$

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c) Compute $A^9 - 6A^8 + 10A^7 - 3A^6 + A + I$

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Where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ - & 0 & 2 \end{bmatrix}$$

Ans:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ - & 0 & 2 \end{bmatrix}$$

02

Characteristic equation is:

$$[A - \lambda I][X] = [0]$$

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$$\lambda^3 - 6\lambda^2 + 10\lambda - 3 = 0$$

By Cayley Hamilton theorem,

04

$$A^3 - 6A^2 + 10A - 3 = 0$$

$$\text{SO, } A^9 - 6A^8 + 10A^7 - 3A^6 + A + I$$

$$= A^6(A^3 - 6A^2 + 10A - 3) + A + I$$

06

$$= A^6(0) + A + I$$

$$= A + I$$

$$= \begin{bmatrix} 2 & 2 & 3 \\ -1 & 4 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

07

Q. 5) a) Evaluate the following integral by using Laplace transform

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$$I = \int_0^{\infty} e^{-t} \left[t \int_0^t e^{-4u} \cdot \cos u du \right] dt$$

Ans:

$$L(\cos t) = \frac{s}{s^2+1}$$

02

$$L(e^{-4t} \cos t) = \frac{s+4}{(s+4)^2+1}$$

$$L(e^{-4t} \cos t) = \frac{s+4}{(s+4)^2+1}$$

$$L\left(\int_0^t e^{-4u} \cos u du\right) = \frac{1}{s} \left[\frac{s+4}{s^2+8s+17} \right]$$

04

$$L\left(t \int_0^t e^{-4u} \cos u du\right) = -\frac{d}{ds} \left[\frac{1}{s} \left[\frac{s+4}{s^2+8s+17} \right] \right]$$

$$= - \left[\frac{(s^2+8s^2+17s)(1)-(s+4)(3s^2+16s+17)}{(s^2+8s^2+17s)^2} \right]$$

Sub. S = 1,

$$= - \left(\frac{25-180}{(26)^2} \right) = 0.22$$

06

b) **Evaluate:**

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$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} dx, a > 0, b > 0.$$

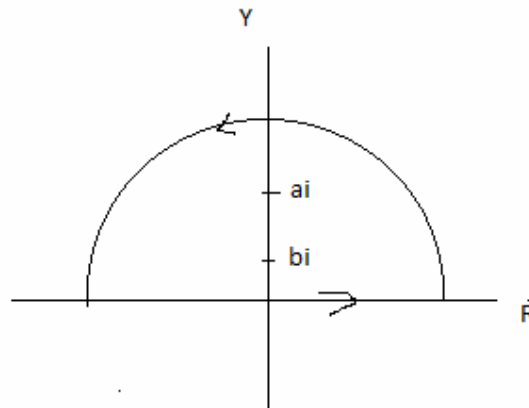
Ans:

$$I = \int_{-\infty}^{\infty} \frac{\cos z}{(z^2 + a^2)(z^2 + b^2)} dz$$

02

Here poles of the integral are:

$$Z = ai, -ai, bi, -bi$$



The poles ai and bi lies inside the close curve C

$$\text{Residue of } f(z) \text{ at } z = ai = \lim_{z \rightarrow ai} (z - ai) \frac{\cos z}{(z - ai)(z + ai)(z^2 + a^2)}$$

$$= \frac{\cosh a}{(2ai)(-a^2+b^2)}$$

$$= \frac{\cosh a}{(2ai)(-a^2+b^2)}$$

05

Residue of $f(z)$ at $z = bi = \lim_{z \rightarrow bi} (z - bi) \frac{\cos z}{(z-bi)(z+bi)(z^2+b^2)}$

$$= \frac{\cos bi}{(2bi)(-b^2+a^2)}$$

06

$$= \frac{\cosh b}{(2bi)(-b^2+a^2)}$$

By Residue theorem,

$$I = 2\pi i \text{ (sum of the residues)}$$

$$= 2\pi i \left[\frac{\cosh a}{(2ai)(-a^2+b^2)} + \frac{\cosh b}{(2bi)(-b^2+a^2)} \right]$$

07

$$= \frac{\pi}{a^2-b^2} \left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right]$$

c) State Cauchy's Integral formula and hence evaluate

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$$\int_{C:|z|=4} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)} dz$$

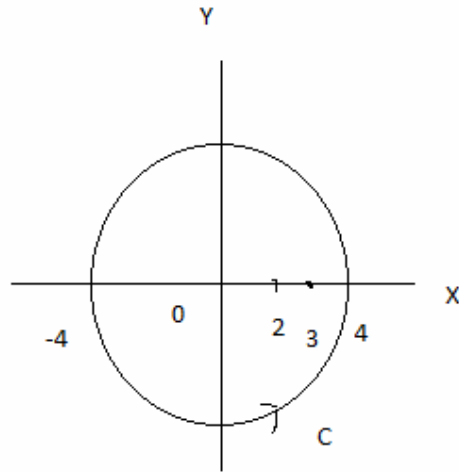
Ans: Statement: If $f(z)$ is analytic everywhere except at $z = a$, then

02

$$\int \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

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Here $z = 2$ and $z=3$ are the poles which lie inside C



$$\frac{1}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$$

04

$$1 = A(z-3) + B(z-2)$$

Solving we get, $A = -1$ and $B = 1$

$$\frac{1}{(z-2)(z-3)} = \frac{-1}{z-2} + \frac{1}{z-3}$$

$$\int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)} dz = \int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-3)} dz - \int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)} dz$$

06

$$= 2\pi i f(3) - 2\pi i f(2)$$

$$= 2\pi i (-1) - 2\pi i (1) = -4\pi i$$

07

Q. 6) a) Find all possible Laurent's expansions of the function

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$$f(z) = \frac{7z-2}{z(z-2)(z+1)} \text{ about } z = -1.$$

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Ans:
$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1} \quad 01$$

$$= \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{(z+1)-1} + \frac{2}{(z+1)-3} - \frac{3}{z+1} \quad 02$$

(i) When $|z+1| < 1$ we write,

$$F(z) = \frac{-3}{z+1} - \frac{1}{1-(z+1)} - \frac{2}{3-(z+1)}$$

When $|z+1| < 1$ then obviously $|z+1| < 3$

$$F(z) = \frac{-3}{z+1} - \frac{1}{1-(z+1)} - \frac{2}{\frac{2}{3}(1-\frac{z+1}{3})}$$

$$= \frac{-3}{z+1} - [1 + (z+1) + (z+1)^2 + \dots] - \frac{2}{\frac{2}{3}} [1 + \frac{z+1}{3} + \frac{z+1^2}{9} + \dots]$$

04

(ii) $1 < |z+1| < 3$

$$F(z) = \frac{-3}{1+z} + \frac{1}{(z+1)(1-\frac{1}{z+1})} - \frac{2}{3(1-\frac{z+1}{3})}$$

$$= \frac{-3}{1+z} + \frac{1}{(z+1)} [1 + \frac{1}{z+1} + \frac{1}{z+1^2} + \dots] - \frac{2}{3} [1 + \frac{z+1}{3} + \frac{z+1^2}{9} + \dots]$$

(iii) $|z+1| > 3$

$$F(z) = \frac{-3}{1+z} + \frac{1}{(z+1)(1-\frac{1}{z+1})} - \frac{2}{(z+1)(1-\frac{3}{z+1})}$$

$$= \frac{-3}{1+z} + \frac{1}{(z+1)} [1 + \frac{1}{z+1} + \frac{1}{z+1^2} + \dots] - \frac{2}{z+1} [1 + \frac{3}{z+1} + \frac{9}{z+1^2} + \dots] \quad 06$$

b) Solve the following equation by using Laplace transform 07

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$$\frac{dy}{dt} + 2y + \int_0^t y dt = \sin t$$

given $y(0)=1$.

02

Ans: Taking laplace transform on both sides,

$$L(y') + 2L(y) + L\left(\int_0^t y dt\right) = L(\sin t)$$

$$s\bar{y} - 1 + 2\bar{y} + \frac{1}{s}\bar{y} = \frac{1}{s^2+1}$$

04

$$\left(s + 2 + \frac{1}{s}\right)\bar{y} = \frac{1}{s^2+1} + 1 = \frac{s^2+1+1}{s^2+1}$$

$$\bar{y} = \frac{s(s^2+2)}{(s+1)^2(s^2+1)}$$

05

$$\text{So, } y = L^{-1}\left(\frac{s(s^2+2)}{(s+1)^2(s^2+1)}\right)$$

$$= L^{-1}\left[\frac{1}{s+1} - \frac{3}{2} \frac{1}{(s+1)^2} + \frac{1}{2} \frac{1}{(s^2+1)}\right]$$

$$Y = e^{-t} - \frac{3}{2}te^{-t} + \frac{1}{2}\sin t$$

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c) Find the bilinear transformation which maps $z=0, 1, -2i$ onto $w = -4i, \infty, 0$ respectively. Also obtain fixed points of the transformation.

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Ans: $w = \frac{at+b}{ct+d}$

01

$$Z = 0, 1, -2i \text{ and } w = -4i, \infty, 0$$

$$-4i = \frac{b}{d}, \frac{1}{0} = \frac{a+b}{c+d}, \quad 0 = \frac{-2ai+b}{-2ci+d}$$

Solving,

$$B = -4di, d = -ci$$

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$B = -4c, a = 2ci$

03

$$w = \frac{ai + b}{ci + d}$$

$$= \frac{2ci - 4c}{cs - ci} = \frac{2iz - 4}{s - i} = -\frac{(2s + 4i)}{is + 1}$$

05

Sub. $w = z$

$$Z = -\frac{(2s + 4i)}{is + 1}$$

So, $z = -4/i, -i$ are fixed points.

07

Q. 7) a) Show that $u = e^{-2xy} \sin(x^2 - y^2)$ is harmonic. Find the corresponding analytic function $f(z)$ and also find the harmonic conjugate v .

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Ans: Let $u = e^{-2xy} \sin(x^2 - y^2)$

01

So, $u_{xx} + u_{yy} = 0$

U is Harmonic.

$$F'(z) = u_x + iv_x$$

02

$$= u_x - iu_y$$

$$F(z) = \int u_x - iu_y$$

By milne Thomson's method,

04

$$F(z) = -ie^{-iz^2} + c$$

So, $v = -e^{-2xy} \cos(x^2 - y^2)$

06

b) Find $L(\operatorname{erf} \sqrt{t})$ and hence find $L(t \operatorname{erf} 2\sqrt{t})$

07

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Ans:

$$erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx$$

$$so, erf(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2} dx$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \left(1 - x^2 + \frac{x^4}{2!} - \dots \right) dx \tag{02}$$

$$= \frac{2}{\sqrt{\pi}} \left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5 \cdot 2!} - \dots \right] \tag{03}$$

so,

$$L(erf(\sqrt{t})) = \frac{1}{s^{3/2}} \left[1 - \frac{1 \cdot 1}{2s} + \frac{1 \cdot 3 \cdot 1}{2 \cdot 4 \cdot s^2} - \dots \right]$$

$$= \frac{1}{s^{3/2}} (1 + 1/s)^{-1/2} \tag{05}$$

$$= \frac{1}{s\sqrt{s+1}}$$

$$L(erf(2\sqrt{t})) = L(erf \sqrt{4t})$$

$$= \frac{1}{4} \frac{1}{s/4 \sqrt{s/4+1}}$$

$$= \frac{2}{s\sqrt{s+4}} \tag{06}$$

$$L(terf 2\sqrt{t}) = -\frac{d}{ds} \left(2(s^3 - 4s^2)^{-1/2} \right)$$

$$= -2 \left(-\frac{1}{2} \right) (s^3 - 4s^2)^{-3/2} (3s^2 + 8s)$$

$$= \frac{3s+8}{s^2(s+4)^{3/2}} \tag{07}$$

c)

If $N = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, then show that $(I-N)(I+N)^{-1}$ is a unitary matrix. 07

Ans:

$$I-N = \begin{bmatrix} 1 & -(1+2i) \\ 1 & 1 \end{bmatrix} \tag{02}$$

03

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$$I + N = \begin{bmatrix} 1 & (1+2i) \\ -1+2i & 1 \end{bmatrix}$$

04

$$(I + N)^{-1} = \frac{1}{6} \begin{bmatrix} 1 & -(1+2i) \\ 1-2i & 1 \end{bmatrix}$$

$$A = (I - N)(I + N)^{-1} = \frac{1}{6} \begin{bmatrix} -4 & -2(1+2i) \\ 2(1-2i) & -4 \end{bmatrix}$$

06

$$\bar{A} = \frac{1}{6} \begin{bmatrix} -4 & -2(1-2i) \\ 2(1+2i) & -4 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} -2 & -(1+2i) \\ (1-2i) & -2 \end{bmatrix}$$

$$A^{\theta} = \frac{1}{3} \begin{bmatrix} -2 & (1+2i) \\ -(1-2i) & -2 \end{bmatrix}$$

07

So we get, $AA^{\theta} = I$