Paper Solution

Subject: Applied Mathematics-IV

Exam: S.E.Comp. Sem-IV (Rev.) Summer-2010

1.

a) $u=\frac{1}{2}\log(x^2+y^2)$, $v=\tan^{-1}\frac{kx}{y}$

f(z) is analytic.

$u_x=v_y$ and $u_y=-v_x$

$$\frac{x}{x^2+y^2}=\frac{-kx}{k^2x^2+y^2}$$

$\Rightarrow -k=1$

$\Rightarrow k=1$

[05]

b)

$$A=\begin{bmatrix}\Pi & \Pi/4\\ 0 & \Pi/2\end{bmatrix}$$

$\lambda=\Pi, \Pi/2$

$\cos A=\alpha_1 A+\alpha_0 I$

$\cos\lambda=\alpha_1\lambda+\alpha_0$

For $\lambda=\Pi, -1=\alpha_1\Pi+\alpha_0$

For $\lambda=\Pi/2, 0=\alpha_1\Pi/2+\alpha_0$

$\therefore \alpha_1=\frac{-2}{\Pi}, \alpha_0=1$

$$\therefore \cos A=\begin{bmatrix}-1 & -1/2\\ 0 & 0\end{bmatrix}$$

[05]

c)

m=2,m+n=4, n=2

To get basic solution we equate n=2 variables to zero.

No. of basic solution=6

No. of basic solution	Non-basic variables	Basic variables	Equations	Feasible?	Degenerate?
1	$x_3 = 0$ $x_4 = 0$	x_1, x_2	$x_1 + 2x_2 = 7$ $2x_1 - x_2 = 4$ $x_1 = 3, x_2 = 2$	Yes	No
2	$x_2 = 0$ $x_4 = 0$	x_1, x_3	$x_1 + x_3 = 7$ $2x_1 + 3x_3 = 4$ $x_1 = -1, x_3 = 2$	No	No
3	$x_1 = 0$ $x_4 = 0$	x_2, x_3	$2x_2 + 4x_3 = 7$ $-x_2 + 3x_3 = 4$ $x_2 = 0.5, x_3 = 1.5$	Yes	No
4	$x_2 = 0$ $x_3 = 0$	x_1, x_4	$x_1 + x_4 = 7$ $2x_1 - 2x_4 = 4$ $x_2 = 4.5, x_4 = 2.5$	Yes	No
5	$x_1 = 0$ $x_3 = 0$	x_2, x_4	$2x_2 + x_4 = 7$ $-x_2 - 2x_4 = 4$ $x_2 = 6, x_4 = -5$	No	No
6	$x_1 = 0$ $x_2 = 0$	x_3, x_4	$4x_3 + x_4 = 7$ $3x_3 - 2x_4 = 4$ $x_3 = 1.6, x_4 = 0.45$	Yes	No

[05]

d)

$$I=\int_{ABC}(z^2+3z)dz$$

$$Along AB:$$

$$x=2, dx=0, y:0\to 2$$

$$dz=dx+idy=idy$$

$$\int_{AB}(z^2+3z)dz=-14+\frac{52}{3}i$$

$$Along BC:$$

$$y=2, dy=0, x:2\to 0$$

$$dz=dx$$

$$\int_{BC}(z^2+3z)dz=-\left(\frac{2}{3}+i20\right)$$

$$\int_{ABC}(z^2+3z)dz=\int_{AB}(z^2+3z)dz+\int_{BC}(z^2+3z)dz$$

$$\int_{ABC}(z^2+3z)dz=-\frac{44}{3}+32i$$

[05]

2.

a)

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda = 0, 3, 15$$

$$for\, \lambda = 0$$

$$(A - \lambda I)X = 0$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$for\, \lambda = 3$$

$$(A - \lambda I)X = 0$$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$for\, \lambda = 15$$

$$(A - \lambda I)X = 0$$

$$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

X_1, X_2, X_3 are linearly independent.

[06]

2-b)

$$f(z)=u+iv$$

$$if(z)=iu-v$$

$$(1+i)f(z)=(u-v)+i(u+v)$$

$$F(z)=U+iV$$

$$\therefore F'(z)=U_x+iV_x$$

$$\therefore F'(z)=V_y+iV_x$$

$$V=u+v=\frac{\sin 2x}{\cosh 2y-\cos 2x}$$

$$V_x=\frac{2\cosh 2y.\cos 2x-2}{(\cosh 2y-\cos 2x)^2}=\Psi_1(x,y)$$

$$V_y=\frac{2\sinh y.\sin 2x}{(\cosh 2y-\cos 2x)^2}=\Psi_2(x,y)$$

$$F'(z)=i\,\frac{2(\cos 2z-1)}{(1-\cos 2z)^2}$$

$$F(z)=i.\cot z+c$$

$$\therefore (1+i)f(z)=i.\cot z+c$$

$$\therefore f(z)=\frac{i}{(1+i)}\cot z+c$$

[05]

2-c)

Max.

$$z-x_1+x_2-3x_3+0s_1+0s_2+0s_3=0$$

$$x_1+x_2+x_3+s_1+0s_2+0s_3=10$$

$$2x_1+0x_2-x_3+0s_1+s_2+0s_3=3$$

$$2x_1-2x_2+3x_3+0s_1+0s_2+s_3=0$$

$$x_1,x_2,x_3,s_1,s_2,s_3\geq 0$$

Iteration No.	Basic variables	Coefficients of						RHS Solution	Ratio
		x_1	x_2	x_3	s_1	s_2	s_3		
0	z	-1	1	-3	0	0	0	0	
s_1 leaves	s_1	1	1	1	1	0	0	10	10
x_3 enters	s_2	2	0	-1	0	1	0	3	-3
	s_3	2	-2	3	0	0	1	0	0
1	z	2	4	0	3	0	0	30	
	x_3	1	1	1	1	0	0	10	
	s_2	3	**1**	0	1	1	0	13	
	s_3	-1	-5	0	-3	0	1	-30	

$$\therefore Z_{\max} = 30, x_1 = 0, x_2 = 0, x_3 = 10$$ **[07]**

3. **[06]**

a)

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\lambda = 2, 3, 6$$

$$For\ \lambda = 2, X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$For\ \lambda = 3, X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$For\ \lambda = 6, X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

3- b)

$$z = x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 - MA_3 ......(1)$$

$$3x_1 + x_2 + s_1 + 0s_2 + 0s_3 + 0A_3 = 3.........(2)$$

$$2x_1 + 3x_2 + 0s_1 + s_2 + 0s_3 + 0A_3 = 6......(3)$$

$$4x_1 + 5x_2 + 0s_1 + 0s_2 + s_3 + A_3 = 20......(4)$$

$$eq.^n (1) + M \square eq.^n (4),$$

$$z + 20M = (1+4M)x_1 + (4+5M)x_2 + 0s_1 + 0s_2 - Ms_3$$

$$z - (1+4M)x_1 - (4+5M)x_2 + 0s_1 + 0s_2 + Ms_3 = -20M ......(5)$$

Iteration No.	Basic variables	Coefficients of x_1	x_2	s_1	s_2	s_3	A_3	RHS Solution	Ratio
0	z	$-(1+4M)$	$-(4+5M)$	0	0	M	0	-20M	
	s_1	3	1	1	0	0	0	3	3
s_2 leaves	s_2	2	**3**	0	1	0	0	6	2
x_2 enters	A_3	4	5	0	0	-1	-1	20	4

0	z	$-(1+4M)$	$-(4+5M)$	0	0	M	0	-20M	
	s_1	3	1	1	0	0	0	3	3
	s_2	2	**3**	0	1	0	0	6	2
	A_3	4	5	0	0	-1	-1	20	4

3-c)

i) Cauchy's Residue theorem:

$$\int_c f(z)dz = 2\Pi i \Sigma \operatorname{Re}s f(z)$$

$$I = \int_c \frac{z-1}{z^2+2z+5}dz$$

$$f(z) = \frac{z-1}{z^2+2z+5}$$

$$Poles:$$

$$z^2+2z+5=0$$

$$z=-1\pm 2i$$

$$\therefore \int_c \frac{z-1}{z^2+2z+5}dz = 0$$

[03]

3-c)

ii)

$$\int_0^{2\Pi} \frac{d\theta}{5-3\cos\theta} = \int_c \frac{1}{5-3\left(\frac{z^2+1}{2z}\right)}.\frac{dz}{iz}$$

$$\int_0^{2\Pi} \frac{d\theta}{5-3\cos\theta} = \frac{2}{i}\int_c \frac{1}{-3z^2+10z-3}dz$$

$$(z-3)(1-3z)=0$$

$$z=3,\frac{1}{3}$$

$$\operatorname{Res} f(z)/_{z=\frac{1}{3}} = \lim_{z\to 1/3}(z-\frac{1}{3})\, f(z)$$

$$\operatorname{Res} f(z)/_{z=\frac{1}{3}} = \frac{1}{4i}$$

$$\therefore \int_0^{2\Pi} \frac{d\theta}{5-3\cos\theta} = 2\Pi i\left[\operatorname{Res} f(z)/_{z=\frac{1}{3}}\right] = \frac{\Pi}{2}$$

[04]

4-a)

$$f(z)=\begin{cases}\dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, z\neq 0\\ 0, z=0\end{cases}$$

$$u=\frac{x^3-y^3}{x^2+y^2}, v=\frac{x^3+y^3}{x^2+y^2}$$

$$\frac{\partial u}{\partial x}=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}=\lim_{x\to 0}\frac{x-0}{x}=1$$

$$\frac{\partial u}{\partial y}=\lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y}=\lim_{y\to 0}\frac{-y-0}{y}=-1$$

$$\frac{\partial v}{\partial x}=\lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x}=\lim_{x\to 0}\frac{x-0}{x}=1$$

$$\frac{\partial v}{\partial y}=\lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y}=\lim_{y\to 0}\frac{y-0}{y}=1$$

$$\therefore \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=1 and \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=-1$$

Hence, C-R equations are satisfied at the origin.
TPT-f’(0) does not exist.

$$f'(z)/_{z=z_0} = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$

$$z \longrightarrow 0 \, along \, y = mx$$

$$f'(0) = \frac{(1-m^3)+i(1+m^3)}{(1+m^2)(1+im)}$$

Since limit depend on path, f’(0) does not exist.

[06]

4-b)

$$f(z) = \frac{z-1}{z^2-2z-3} = \frac{z-1}{(z-3)(z+1)} = \frac{1}{2}\frac{1}{(z-3)} + \frac{1}{2}\frac{1}{(z+1)}$$

$$Poles:$$

$$z = 3 \,\&\, z = -1$$

$$|z| = 3 \,\&\, |z| = 1$$

$$i) For\, |z| < 1$$

$$f(z) = -\frac{2}{9}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+.....\right]+\frac{1}{2}\left[1-z+z^2-z^3+.......\right]$$

$$ii) For\, 1 < |z| < 3$$

$$f(z) = -\frac{2}{9}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+.....\right]+\frac{1}{2z}\left[1-\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+.......\right]$$

$$ii) For\, |z| > 3$$

$$f(z) = \frac{2}{3z}\left[1+\left(\frac{3}{z}\right)+\left(\frac{3}{z}\right)^2+.....\right]+\frac{1}{2z}\left[1-\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+.......\right]$$

[07]

5-

a)

$$f(x_1, x_2, x_3) = x_1 + 2x_3 + x_2 x_3 - x_1^2 - x_2^2 - x_3^2$$

$$\frac{\partial f}{\partial x_1} = 1 - 2x_1 = 0 \Longrightarrow x_1 = \frac{1}{2}$$

$$\frac{\partial f}{\partial x_2} = x_3 - 2x_2 = 0$$

$$\frac{\partial f}{\partial x_3} = 2 + x_2 - 2x_3 = 0$$

$$\therefore x_2 = \frac{2}{3}, x_3 = \frac{4}{3}$$

$$\therefore X_0 = \left(\frac{1}{2}, \frac{2}{3}, \frac{4}{3}\right)$$

$$H = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$D_1 = -2, D_2 = 4, D_3 = -6$$

$$\therefore Z_{\max} = \frac{19}{12}$$

[06]

5- b)

$$z_1 = 1, z_2 = -i, z_3 = 2$$

$$w_1 = 0, w_2 = 2, w_3 = -i$$

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

$$w = \frac{2(z-1)}{(1+i)z - 2}$$

[07]

6-

a)

$$|A - \lambda I| = \lambda^2 - 3\lambda - 2$$

$$A^2 - 3A - 2I = 0$$

$$2A^4 - 5A^3 - 7A + 6I = 16A + 20I = 16\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} + 20\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 36 & 32 \\ 32 & 52 \end{bmatrix}$$

b)

$$i) I = \int_c \frac{\sin z}{4z^2 - 8iz} dz$$

$$f(z) = \frac{\sin z}{4z^2 - 8iz} = \frac{\sin z}{4z(z-2i)}$$

$$Poles:$$

$$z = 0, z = 2i = (0,2)$$

$$\operatorname{Re} sf(z)/_{z=0} = \lim_{z \to 0} z. \frac{\sin z}{4z(z-2i)} = 0$$

$$\operatorname{Re} sf(z)/_{z=2i} = \lim_{z \to 2i} (z-2i). \frac{\sin z}{4z(z-2i)} = \frac{\sin 2i}{8i}$$

$$\therefore I = 2\Pi i \Sigma \operatorname{Re} sf(z) = \frac{\Pi}{4}.\sin 2i$$

$$ii) I = 2\Pi i \Sigma \operatorname{Re} sf(z) = 0$$

6-c) **[07]**

$$f(x_1, x_2) = 2x_1 + 3x_2 - x_1^2 - 2x_2^2$$

$$h_1(x_1, x_2) = x_1 + 3x_2 - 6$$

$$h_2(x_1, x_2) = 5x_1 + 2x_2 - 10$$

$$L(x_1, x_2, \lambda_1, \lambda_2) = f(x_1, x_2) - \lambda_1 h_1(x_1, x_2) - \lambda_2 h_2(x_1, x_2)$$

$$L = (2x_1 + 3x_2 - x_1^2 - 2x_2^2) - \lambda_1(x_1 + 3x_2 - 6) - \lambda_2(5x_1 + 2x_2 - 10)$$

$$\frac{\partial L}{\partial x_1} = 2 - 2x_1 - \lambda_1 - 5\lambda_2 = 0$$

$$\frac{\partial L}{\partial x_1} = 3 - 4x_2 - 3\lambda_1 - 2\lambda_2 = 0$$

$$\lambda_1 h_1(x_1, x_2) = \lambda_1(x_1 + 3x_2 - 6) = 0$$

$$\lambda_2 h_2(x_1, x_2) = \lambda_2(5x_1 + 2x_2 - 10) = 0$$

$$h_1(x_1, x_2) = (x_1 + 3x_2 - 6) \geq 0$$

$$h_2(x_1, x_2) = (5x_1 + 2x_2 - 10) \geq 0$$

$$x_1, x_2, \lambda_1, \lambda_2 \geq 0$$

$$Case i) \lambda_1 = 0, \lambda_2 = 0$$

$$x_1 = 1, x_2 = \frac{3}{4}$$

$$z = \frac{17}{8}$$

$$Case ii) \lambda_1 = 0, \lambda_2 \neq 0$$

$$x_1 = 1.648, x_2 = 0.88$$

$$z = 1.6713$$

$$Case iii) \lambda_1 \neq 0, \lambda_2 = 0$$

$$x_1 = \frac{3}{2}, x_2 = \frac{3}{2}$$

$$reject.$$

$$Case iv) \lambda_1 \neq 0, \lambda_2 \neq 0$$

$$x_1 = 1.385, x_2 = 1.538$$

$$\lambda \quad 1.09$$

7-a)

$$u = r^2 \cos 2\theta = \alpha$$

By CR equations,

$$\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta}$$

$$2r \cos 2\theta = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial \theta} = 2r^2 \cos 2\theta$$

$$\therefore v = 2r^2 \frac{\sin 2\theta}{2} + c$$

$$v = r^2 \sin 2\theta$$

Orthogonam trajectory is,

$$r^2 \sin 2\theta = \beta$$

[06]

7-b)

$$u = r^2 \cos 2\theta = \alpha$$

$$w = z + \frac{1}{z}$$

$$put z = re^{i\theta}, w = u + iv$$

$$u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

$$put \theta = \frac{\Pi}{3} \&$$

$$Solving,$$

$$\frac{u^2}{2} - \frac{v^2}{3} = 1$$

7-c)

$$f(x_1,x_2)=6x_1+8x_2-x_1^{\;2}-x_2^{\;2}$$
$$h_1(x_1,x_2)=4x_1+3x_2-16$$
$$h_2(x_1,x_2)=3x_1+5x_2-15$$
$$L(x_1,x_2,\lambda_1,\lambda_2)=(6x_1+8x_2-x_1^{\;2}-x_2^{\;2})-\lambda_1\left(4x_1+3x_2-16\right)-\lambda_2\left(3x_1+5x_2-15\right)$$
$$\frac{\partial L}{\partial x_1}=6-2x_1-4\lambda_1-3\lambda_2=0$$
$$\frac{\partial L}{\partial x_2}=8-2x_2-3\lambda_1-5\lambda_2=0$$
$$\frac{\partial L}{\partial \lambda_1}=-(4x_1+3x_2-16)=0$$
$$\frac{\partial L}{\partial x_2}=-(3x_1+5x_2-15)=0$$
$$x_1=\frac{35}{11},x_2=\frac{12}{11}$$
— $$X_0=\left(\frac{35}{11},\frac{12}{11}\right)$$
$$H_0^B=\begin{bmatrix}0 & P\\ P' & Q\end{bmatrix}$$
$$P=\begin{bmatrix}4 & 3\\ 3 & 5\end{bmatrix},Q=\begin{bmatrix}-2 & 0\\ 0 & -2\end{bmatrix}$$
$$H_0^B=\begin{bmatrix}0 & 0 & 4 & 3\\ 0 & 0 & 3 & 5\\ 4 & 3 & -2 & 0\\ 3 & 5 & 0 & -2\end{bmatrix}$$
$$\left|H_0^B\right|=121$$
$$\because n=2,m=2$$
$$\therefore n-m=0$$

$\therefore$ The above method fails. **[07]**