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Mechanical Vibration

Q1 a) *Find the natural frequency of the system shown in Fig. The cord may be assumed in extensible in the spring mass pulley system and no slip.* *10*

S ol^n: Total energy= K.E. of mass(mass)+K.E. of pulley+P. E. stored in spring

$$T=\frac{1}{2}m_1\dot{x}^2+\frac{1}{2}m\dot{y}^2+\frac{1}{2}J\dot{\theta}^2 -----(1)$$

For any small displacement θ,

x=r θ (where x=displacement of mass at any instant)-----------

y=$x/2$ (y = Vertical displacement of pulley center)------------- (2)

and J=1/2mr^2

For pulley

Put equation no(2) in equation no (1)

$$T=\frac{1}{2}m_1\dot{x}^2+\frac{1}{2}m\left(\frac{\dot{x}}{2}\right)^2+\frac{1}{2}\frac{1}{2}mr^2\frac{\dot{x}^2}{r^2}$$

$$T=\frac{1}{2}m_1\dot{x}^2+\frac{1}{2}m\left(\frac{\dot{x}}{2}\right)^2+\frac{1}{2}\frac{1}{2}mr^2\dot{\theta}^2$$

$$=\frac{1}{2}m_1\dot{x}^2+\frac{3}{8}m\dot{x}^2 -----(3)$$

The potential energy is given by

$$V=P.E.=\frac{1}{2}ky^2=\frac{1}{2}k\left(\frac{1}{2}x\right)^2=\frac{1}{2}kx^2 ----(4)$$

The energy of the system is constant

$$\frac{d}{dt}(T+V)=0$$

$$\frac{d}{dt}\left(\frac{1}{2}m_1\dot{x}^2+\frac{3}{8}m\dot{x}^2+\frac{1}{8}kx^2\right)=0$$

$$m_1\dot{x}\ddot{x}+\frac{3}{4}m\dot{x}\ddot{x}+\frac{1}{4}kx\dot{x}=0$$

$$m_1\ddot{x}+\frac{3}{4}m\ddot{x}+\frac{1}{4}kx=0$$

$$\left(m_1+\frac{3}{4}m\right)\ddot{x}+\frac{1}{4}kx=0----(5)$$

So,natural frequency is

$$\omega_n=\sqrt{\frac{1}{4}\frac{k}{\left(m_1+\frac{3m}{4}\right)}}$$

$$\omega_n=\sqrt{\frac{k}{(4m_1+3m)}}---rad/\sec$$

Q.1(b)*Use lagrange's equations to derive the differential equation governing the motion of the system of fig.using X_1 and X_2 as generalized co-ordinaties. Write the differential equation in matrix form.* *10*

Solution: The kinetic energy of the system at an arbitrary instant is

$$T=\frac{1}{2}m\left(\frac{\dot{x}_1+\dot{x}_2}{2}\right)^2+\frac{1}{2}I\left(\frac{\dot{x}_2-\dot{x}_1}{L}\right)^2$$

The potential energy of the system is

$$V=\frac{1}{2}K\left(\frac{3}{4}x_1+\frac{1}{4}x_2\right)^2+\frac{1}{2}kx_2^2$$

The Lagrangian is

L=T-V

$$L=\frac{1}{2}m\left(\frac{\dot{x}_1+\dot{x}_2}{2}\right)^2+\frac{1}{2}I\left(\frac{\dot{x}_2-\dot{x}_1}{L}\right)^2-\frac{1}{2}K\left(\frac{3}{4}x_1+\frac{1}{4}x_2\right)^2-\frac{1}{2}kx_2^2$$

If variation δx_1 and δx_2 are introduced, the work done by the external moment is

$$\delta w=M(t)\delta\left(\frac{x_2-x_1}{L}\right)=-M(t)\delta x_1+\frac{1}{L}M(t)\delta x_2$$

Application of Lagranges equations leads to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\frac{\partial L}{\partial x_1}=Q_1$$

$$\frac{d}{dt}\left[m\left(\frac{\dot{x}_1+\dot{x}_2}{2}\right)\left(\frac{1}{2}\right)+I\left(\frac{\dot{x}_2-\dot{x}_1}{L}\right)\left(-\frac{1}{L}\right)\right]+K\left(\frac{3}{4}x_1+\frac{1}{4}x_2\right)\left(\frac{3}{4}\right)=-\frac{1}{L}M(t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)-\frac{\partial L}{\partial x_2}=Q_2$$

$$\frac{d}{dt}\left[m\left(\frac{\dot{x}_1+\dot{x}_2}{2}\right)\left(\frac{1}{2}\right)+I\left(\frac{\dot{x}_2-\dot{x}_1}{L}\right)\left(\frac{1}{L}\right)\right]+\left[k\left(\frac{3}{4}x_1+\frac{1}{4}x_2\right)\left(\frac{1}{4}\right)+kx_2\right]=\frac{1}{L}M(t)$$

Rearranging and writing in matrix form leads to

$$\begin{bmatrix}\frac{m}{4}+\frac{I}{L^2} & \frac{m}{4}-\frac{I}{L^2}\\ \frac{m}{4}-\frac{I}{L^2} & \frac{m}{4}+\frac{I}{L^2}\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix}+\begin{bmatrix}\frac{9}{16}k & \frac{3}{16}k\\ \frac{3}{16}k & \frac{17}{16}k\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}-\frac{1}{L}M(t)\\ \frac{1}{L}M(t)\end{bmatrix}$$

Q:2(a) *Determine the steady-state amplitude of angular oscillation for the system.* *10*

Solution: Free body diagram of the system at an arbitrary instant are shown in Fig. Summing moments about o,

$$-k\left(\frac{3}{4}L\theta-Y\right)\left(\frac{3}{4}L\right)-\frac{1}{4}CL\dot{\theta}\left(\frac{L}{4}\right)=\frac{1}{12}mL^2\ddot{\theta}+\frac{1}{4}mL\ddot{\theta}\left(\frac{1}{4}L\right)$$

$$\frac{7}{48}mL^2\ddot{\theta}+\frac{1}{16}cL^2\dot{\theta}+\frac{9}{16}kL^2\theta=\frac{3}{4}kLy(t)$$

$$\frac{7}{48}mL^2\ddot{\theta}+\frac{1}{16}cL^2\dot{\theta}+\frac{9}{16}kL^2\theta=\frac{3}{4}kLY\sin\omega t---(1)$$

$$m\ddot{x}+c\dot{x}+kx=F_O\sin\omega t----(2)$$

Where $F_O=\frac{3}{4}kLY$

The natural frequency and damping ratio are

$$\omega_n = \sqrt{\frac{\frac{9}{16}kL^2}{\frac{7}{48}mL^2}} = \sqrt{\frac{27k}{7m}}$$

$$\therefore \omega_n = \sqrt{\frac{27\times 2\times 10^5}{7(10)}} = 277.8 - -rad/\sec$$

$$m_{equ}\ddot{x} + c_{equ}\dot{x} + k_{equ}x = F_{equ}(t)$$

$$\ddot{x} + \frac{c_{equ}}{m_{equ}}\dot{x} + \frac{k_{equ}}{m_{equ}}x = \frac{1}{m_{equ}}F_{equ}(t)$$

$$\frac{k_{equ}}{m_{equ}} = \omega_n{}^2$$

$$\frac{c_{equ}}{m_{equ}} = \frac{c}{c_c}.\frac{c_c}{m_{equ}} = \zeta.2\omega_n$$

$$\frac{c_c}{2m} = \omega_n$$
$$\frac{c_c}{m} = 2\omega_n$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n{}^2x = \frac{1}{m_{equ}}F_{equ}(t)$$

$$2\zeta\omega_n = \frac{\frac{1}{16}cL^2}{\frac{7}{48}mL^2}$$

$$\zeta = \frac{3c}{14m\omega_n} = \frac{3(400N-s/m)}{14(10kg)(277.7rad/\sec)}$$
$$\zeta = 0.0309$$

Now, comparing equation no(1) with (2),

$$F_O = \frac{3}{4}kLY = \frac{3}{4}(2\times 10^5 N/m)(1.2m)(0.01m)$$
$$F_O = 1800N-m$$
$$m_{equ} = \frac{7}{48}mL^2 = \frac{7}{48}(10kg)(1.2m)^2 = 2.1kg-m^2$$

The frequency ratio for the system is

$$r = \frac{\omega}{\omega_n} = \frac{350 rad/\sec}{277.8 rad/\sec} = 1.26$$

The system magnification factor is

$$M.F = \frac{X}{\Delta_{st}} = \frac{1}{\sqrt{\left[1-r^2\right]^2 + \left[2\zeta r\right]^2}}$$

$$M(1.26, 0.0309) = \frac{1}{\sqrt{\left[1-(1.26)^2\right]^2 + \left[2(0.0309)(1.26)\right]^2}}$$

$$M.F = 1.69$$

The steady-state amplitude is obtained using equation

$$\frac{m_{equ}\omega_n^2\theta}{F_O} = M(1.26, 0.0309)$$

$$\theta = \frac{F_O M(1.26, 0.0309)}{m_{equ}\omega_n^2} =$$

$$\theta = \frac{(1800N-m)(1.69)}{(2.1kg-m^2)(277.8 rad/\sec)^2}$$

$$\theta = 0.0188 rad = 1.08^0$$

Q2(b): *Find the Eigen values and Eigen vectors of the system shown in fig.* (12)

The first step is to write the differential equations of motion for the three masses by Newton's second law of motion:

$$4m\ddot{x}_1 = -3kx_1 - k(x_1 - x_2)$$
$$2m\ddot{x}_2 = k(x_1 - x_2) - k(x_2 - x_3)$$
$$m\ddot{x}_3 = k(x_2 - x_3)$$

These can be rearranged in the following forms:

$$4m\ddot{x}_1 + 4kx_1 - kx_2 = 0$$
$$2m\ddot{x}_2 - kx_1 + 2kx_2 - kx_3 = 0 ------ equ^n(1)$$
$$m\ddot{x}_3 - kx_2 + kx_3 = 0$$

The differential equations of motion for the system written by the method of Newton's second law in equation are rearranged as follows:

$$(4m\ddot{x}_1 + 4kx_1) - kx_2 = 0$$
$$-kx_1 + (2m\ddot{x}_2 + 2kx_2) - kx_3 = 0 ------ equ^n(2)$$
$$-kx_2 + (m\ddot{x}_3 + kx_3) = 0$$

Assume for the principal mode of vibration, the solutions to be

$$x_1 = X_1 \sin \omega t$$
$$x_2 = X_2 \sin \omega t -----(3)$$
$$x_3 = X_3 \sin \omega t$$

Substituting these equations in equation(2) and cancelling out the common factor $\sin \omega t$, gives

$$\left(4k - 4m\omega^2\right)X_1 - kX_2 = 0$$
$$-kX_1 + \left(2k - 2m\omega^2\right)X_2 - kX_3 = 0 -------(4)$$
$$-kX_2 + \left(k - m\omega^2\right)X_3 = 0$$

This further gives,

$$\begin{bmatrix} \left(4k - 4m\omega^2\right) & -k & 0 \\ -k & \left(2k - 2m\omega^2\right) & -k \\ 0 & -k & \left(k - m\omega^2\right) \end{bmatrix} = 0 ----(5)$$

After expanding, we have

$$\left(k - m\omega^2\right)\left(8m^2\omega^4 - 16km\omega^2 + 3k^2\right) = 0 ---------(6)$$

Hence, the three natural frequencies are

$$\omega_1 = 0.457\sqrt{\frac{k}{m}}$$
$$\omega_2 = \sqrt{\frac{k}{m}} -------------(7)$$
$$\omega_3 = 1.338\sqrt{\frac{k}{m}}$$

In order to find the mode shapes, divide two of the equations (4) by X_1 throughout, and rearrange as follows, for i^{th} mode shape.

$$\left(\frac{X_2}{X_1}\right)_i = \frac{4}{k}\left(k - m\omega_i^2\right)$$
$$\left(\frac{X_3}{X_1}\right)_i = \frac{8}{k^2}\left(k - m\omega_i^2\right)^2 - 1 ---------(8)$$

The three mode shapes can now be obtained by substituting in the above equations three values of ω as obtained in expressions (7), one at a time, giving

$\left(\frac{X_2}{X_1}\right)_1 = 3.16$ $\left(\frac{X_3}{X_1}\right)_1 = 4.00$

$\left(\frac{X_2}{X_1}\right)_2 = 0$ $\left(\frac{X_3}{X_1}\right)_2 = -1.00$

$\left(\frac{X_2}{X_1}\right)_3 = 3.16$ $\left(\frac{X_3}{X_1}\right)_1 = 4.00$

Therefore the three mode shapes are

(1 : 3.16 : 4), (1 : 0 : -1) and (1 : -3.16 : 4)

These mode shapes are shown in fig.

Q:3(q) *The mass and stiffness matrices of a vibrating system are given by-*

$$[m] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad [k] = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Where m=1 kg , k=1 N/m

Using holzer's method , determine the two natural frequencies which are above the value 0.8 rad/sec , and draw the mode shapes for the frequencies you have found. 10

Solution: m1=$_1$ m_2=2 m_3=2

K_1=1 k_2=2 k_3=1

Trail no. 1

Assumed value	Sr.No.	Mass(m)	m ω^2	x	$m\omega^2 x$	$\sum m\omega^2 x$	k	$\sum \frac{m\omega^2 x}{k}$
=0.9	1	1	0.81	1	0.81	0.81	1	0.81
=0.81	2	2	1.62	0.19	0.3078	1.1178	2	0.5581
	3	2	1.62	-0.3689	-0.5976	0.52018	1	0.52018
				-0.889				

Trail no. 2

	1	1	1.21	1	1.21	1.21	1	1.21
=1.1	2	2	2.42	-0.21	-0.5082	0.7018	2	0.3509

=1.21	3	2	2.42	-0.5609	-1.3573	-0.655	1	-0.655
				0.0941				

Approximately equal to zero.

$\therefore \omega = 1.1$ is first natural frequency of the system.

Trail no.3

	1	1	1.44	1	1.44	1.44	1	1.44
=1.2	2	2	2.88	-0.44	-1.2672	0.1728	2	0.0864
=1.44	3	2	2.88	-0.5264	-1.516	-1.343	1	-1.343
				0.8166				

Trail no.4

	1	1	2.56	1	2.56	2.56	1	2.56
=1.6	2	2	5.12	-1.56	-7.987	-5.427	2	-2.7136
=2.56	3	2	5.12	1.1536	5.9064	0.4794	1	0.4794
				0.6742				

Trail no.5

	1	1	2.7925	1	2.7225	2.7225	1	2.7225
=1.65	2	2	5.445	-1.7225	-9.379	-6.656	2	-3.228
=2.7225	3	2	5.445	1.6055	8.742	2.0559	1	2.085
				-0.4795				

Trail no.6

	1	1	2.6569	1	2.6569	2.6569	1	2.6569

=1.63	2	2	5.3138	-1.6569	-8.804	-6.1475	2	-3.073
=2.6569	3	2	5.3138	1.4161	7.5248	1.377	1	1.377
				0.0391				

Approximately equal to zero.

$\therefore \omega = 1.63$ is second natural frequency of the system.

Thus $\omega = 1.1$ rad/sec and $\omega = 1.63$ rad/sec are the two natural frequencies of the system.

Q:3(b) *A spring-mass-damper ,having an undamped natural frequency of 100 Hz and a damping constant of 20 N-s/m, is used an accelerometer to measure the vibration of a machine operating at a speed of 3000 rpm, if the actual acceleration is 10 m/s^2 and the recorded acceleration is 9 m/s^2 , find the mass and the spring constant of the accelerometer.*

Solution:

Given data:

f_n =100 Hz , $\omega_n = 2\pi f_n = 2\pi$ x100=628.31 rad/sec

C=20 N-s/m , N=3000 rpm , $\therefore \omega = \frac{2\pi N}{60}$ =314.15 rad/sec

Actual acceleration=10 m/s^2

Recorded acceleration=9 m/s^2

The ratio of measured to true acceleration is given by

$$\frac{1}{\sqrt{\left[\left(1-r^2\right)^2+\left(2\zeta r\right)^2\right]}} = \frac{measured}{true} = \frac{9}{10} = .9$$

$$\left[\left(1-r^2\right)^2+\left(2\zeta r\right)^2\right] = \left(\frac{1}{0.9}\right)^2 = 1.2345$$

The operating speed of the engine gives

$$\therefore \omega = \frac{2\pi N}{60} = 314.15 \text{ rad/sec}$$

The damped natural frequency of vibration of the accelerometer is

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

And the frequency ratio gives,

$$r = \frac{\omega}{\omega_n} = \frac{314.15}{628.31} = 0.499$$

$$\left[(1-r^2)^2 + (2\zeta r)^2\right] = \left(\frac{1}{0.9}\right)^2 = 1.2345$$

$$\left[(1-0.499^2)^2 + (2\zeta 0.499)^2\right] = 1.2345$$

$0.564+4\zeta^2(.249)=1.2345$

$0.996\zeta^2=0.6705$

$\zeta^2=0.673$

$\zeta=0.820$

The suspended mass can be determined from

$C=2m\omega_n\zeta$

20=2m(628.31)(0.820)

$$m = \frac{20}{2(628.31)(0.820)}$$

m =0.019 kg.-------------------ans.

And the spring constant of the accelerometer is given by,

K= $m \cdot \omega_n^2$

= $(0.019)\ (628.31)^2$

= 7500.69 N/m ----------ans.

Q:4 (a) *Proove that for finding the natural frequency of a torsional system the mass moment of inertia of the shaft can be taken in to account by adding one third of its inertia to the disc inertia.* *10*

Ans: Let L be the length of the shaft under equilibrium condition. Consider an element dy of the shaft at a distance y from the support.

If ρ is the mass per unit length of shaft in equilibrium condition, then the mass moment of inertia of shaft $J_S = \rho\, l$ and mass of the element dy is equal to ρ .dy

At any instant, let the disc be displaced from the equilibrium position through a distance θ. Then the potential energy of the system is,

$$\text{P.E.} = \frac{1}{2} K_t.\ \theta^2 \text{ -----------(1)}$$

The k.E. of vibration of the system at this instant consists of K. E. OF the disc plus K. E. of the shaft. The K.E.of the disc is equal to $\frac{1}{2} J\ \dot{\theta}^2$

The K. E. of the element dy of the shaft is $= \frac{1}{2}\ (\rho\,.dy)\left(\frac{y}{l}\dot{\theta}\right)^2$

Therefore the total kinetic energy of the system is given by

$$\text{K. E.} = \frac{1}{2} J\ \dot{\theta}^2 + \int_0^l \frac{1}{2}(\rho.dy)\left(\frac{y}{l}\theta\right)\left(\frac{y}{l}\dot{\theta}\right)^2$$

$$= \frac{1}{2} J\ \dot{\theta}^2 + \frac{1}{2}\ \rho\,.\frac{\dot{\theta}^2}{l^2}.\frac{l^3}{3}$$

$$= \frac{1}{2} J\ \dot{\theta}^2 + \frac{1}{2}\ \frac{J_s}{3}\ \dot{\theta}^2$$

Where $J_s = \rho\,.\,l$ is the mass moment of inertia of shaft.

$$\text{K.E.} = \frac{1}{2}\left(J + \frac{J_s}{3}\right)\ \dot{\theta}^2 \text{ ----------------(2)}$$

Adding equations (1) and (2), we have

$$\frac{1}{2} K_t.\ \theta^2 + \frac{1}{2}\left(J + \frac{J_s}{3}\right)\ \dot{\theta}^2 = \text{constant}$$

Differentiating and cancelling out the common factor $\dot{\theta}$, we have

$$K_t\ \theta + \left(J + \frac{J_s}{3}\right)\ddot{\theta} = 0$$

giving

$$\omega_n = \sqrt{\frac{k_t}{J + \frac{J_s}{3}}} \quad \text{rad/sec}$$

Which shows that for finding the natural frequency of the system , the mass moment of inertia of the shaft can be taken in to account by adding one-third of its inertia to the disc inertia.

Q:4 (b) *A vertical spring mass has a mass of 0.5 kg. and an initial deflection of 2 mm. Find the spring constant and the natural frequeny of the system is subjected in coulomb damping. When displaced from the equilibrium and released it undergoes complete 10 cycles and comes to rest in the extreme position on the side on which it is displaced. Calculate the coulomb damping and the final rest position.* *10*

Solution: Given data:

m=0.5 kg.

δ_{st} = 2 mm = 2×10^{-3} m

X_o = 2 mm

n=10 cycles

Find:

- spring constant (k)
- natural frequency of the system
- coulomb damping
- final rest position.

$$\delta_{st} = \frac{force}{stiffness}$$

$$\therefore k = \frac{mg}{\delta_{st}} = \frac{0.5\times9.81}{2\times10^{-3}}$$

$k = 2452.5$ N/m ------ans.

Natural frequency of the system

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2452.5}{0.5}} = 70.036 \text{ rad/sec}$$

$$f_n = \frac{\omega_n}{2\pi} = 11.147 \text{ Hz.} \quad \text{--------ans.}$$

It is given that the system complets 10 full cycles and stops at the extreme position on the side on which it was displaced. Hence, $X_A = X_B$ In this position, the spring force will be equal to the friction force.

Fspring = k XB = F

Decay, $\Delta = \frac{4F}{k} = \frac{4FX_B}{k} = 4X_B$

The amplitude is to reduce from X_O to X_B in 10 cycles,

$X_B = X_O - n\ \Delta$

$X_B = X_o - n \times 4X_B$

$= \frac{X_O}{4n+1}$

$= \frac{2}{4 \times 10 + 1}$

= 0.04878 mm

This would be the final rest position in this case. The coulomb damping would be,

$F = k\ X_B = 2452.5\ (0.04878 \times 10^{-3})$

F = 0.119 N ----------ans.